An-Najah National University

Faculty of Engineering

Industrial Engineering Department

Course :

Quantitative Methods (65211)

Instructor:

Eng. Tamer Haddad

2nd Semester 2009/2010

Chapter 5

Joint Probability Distributions

- It is often useful to have more than one random variable defined in a random experiment.

- **Example:**

- The continuous random variable X can denote the length of one dimension of an injection-molded part, and the continuous random variable Y might denote the length of another dimension. We might be interested in probabilities that can be expressed in terms of both X and Y.

- In general, if X and Y are two random variables, the probability distribution that defines their simultaneous behavior is called a **joint probability distribution.**

5-1 TWO DISCRETE RANDOM VARIABLES:

5-1.1 Joint Probability Distributions

- **Example:**

Calls are made to check the airline schedule at your departure city.
In the first four bits transmitted:

X: # of bars of signal strength on your cell phone
Y: # of times you need to state your departure city.

We define the range of the random variables (X,Y) to be the sets of points (x,y) in two dimensional space for which the probability that X=x and Y=y is positive.

	x = # of bars of signal strength		
y = # of times city name is stated	1	2	3
4	0.15	0.1	0.05
3	0.02	0.1	0.05
2	0.02	0.03	0.2
1	0.01	0.02	0.25

- Two random variables joint distribution → bivariate probability distribution

- The distribution can be described through a joint probability mass function.

- Also, P(X = x and Y = y) is usually written as P(X = x, Y = y).

❖ Joint Probability Mass Function:

The **joint probability mass function** of the discrete random variables X and Y, denoted as $f_{XY}(x, y)$, satisfies

(1) $f_{XY}(x, y) \geq 0$

(2) $\sum_{x} \sum_{y} f_{XY}(x, y) = 1$

(3) $f_{XY}(x, y) = P(X = x, Y = y)$

5-1.2 Marginal Probability Distributions

❖ The individual probability distribution of a random variable is referred to as its **marginal probability distribution.**

❖ In general, the marginal probability distribution of X can be determined from the joint probability distribution of X and other random variables.

❖ For P(X = x), we sum P(X = x, Y = y) over all points in the range of (X,Y) for which X = x.

➢ Example:

	x = # of bars of signal strength			
y = # of times city name is stated	1	2	3	Marginal probability distribution of Y
4	0.15	0.1	0.05	**0.3**
3	0.02	0.1	0.05	**0.17**
2	0.02	0.03	0.2	**0.25**
1	0.01	0.02	0.25	**0.28**
	0.2	**0.25**	**0.55**	
	Marginal probability distribution of X			

❖ Marginal Probability Mass Function:

If X and Y are discrete random variables with joint probability mass function $f_{XY}(x, y)$, then the **marginal probability mass functions** of X and Y are

$$f_X(x) = P(X = x) = \sum_{R_x} f_{XY}(x, y) \quad \text{and} \quad f_Y(y) = P(Y = y) = \sum_{R_y} f_{XY}(x, y)$$

where R_x denotes the set of all points in the range of (X, Y) for which $X = x$ and R_y denotes the set of all points in the range of (X, Y) for which $Y = y$

E(X) = 1(0.2) + 2(0.25) + 3(0.55) = 2.35

❖ **Same for E(Y), V(X) and V(Y).**

5-1.3 Conditional Probability Distributions

❖ When two random variables are defined in a random experiment, knowledge of one can change the probabilities that we associate with the values of the other.

❖ We can write such conditional probabilities as $P(Y = 1|X = 3)$ and $P(Y=1|X=1)$.

❖ X and Y are expected to be dependent.

➢ **Example:**

For the same example,

$P(Y=1|X=3)\ P(X=3,Y=1)/P(X=3) = f_{xy}(3,1)/f_x(3) = 0.25/0.55 = 0.454$

❖The set of probabilities defines the conditional probability distribution of Y given that X=3

❖ Conditional Probability Mass Function:

Given discrete random variables X and Y with joint probability mass function $f_{XY}(x, y)$ the **conditional probability mass function** of Y given $X = x$ is

$$f_{Y|x}(y) = f_{XY}(x, y)/f_X(x) \qquad \text{for } f_X(x) > 0$$

Because a conditional probability mass function $f_{Y|x}(y)$ is a probability mass function for all y in R_x, the following properties are satisfied:

(1) $f_{Y|x}(y) \geq 0$

(2) $\sum_{R_x} f_{Y|x}(y) = 1$

(3) $P(Y = y | X = x) = f_{Y|x}(y)$

➢ **Example:**

For the same example,

Conditional probability distributions of Y given X = x,

	x = # of bars of signal strength		
y = # of times city name is stated	1	2	3
4	0.75	0.4	0.091
3	0.1	0.4	0.091
2	0.1	0.12	0.364
1	0.05	0.08	0.454

Properties of random variables can be extended to a conditional probability distribution of Y given X = x. The usual formulas for mean and variance can be applied to a conditional probability mass function.

❖ Conditional Mean and Variance:

Let R_x denote the set of all points in the range of (X, Y) for which $X = x$. The **conditional mean** of Y given $X = x$, denoted as $E(Y|x)$ or $\mu_{Y|x}$, is

$$E(Y|x) = \sum_{y} y f_{Y|x}(y)$$

and the **conditional variance** of Y given $X = x$, denoted as $V(Y|x)$ or $\sigma^2_{Y|x}$, is

$$V(Y|x) = \sum_{y} (y - \mu_{Y|x})^2 f_{Y|x}(y) = \sum_{y} y^2 f_{Y|x}(y) - \mu^2_{Y|x}$$

➢ **Example:**

For the same example,

The conditional mean of Y given X = 1, is

$E(Y|1) = 1(0.05) + 2(0.1) + 3(0.1) + 4(0.75) = 3.55$

The conditional variance of Y given X = 1, is

$V(Y|1) = (1-3.55)^2 0.05 + (2-3.55)^2 0.1 + (3-3.55)^2 0.1 + (4-3.55)^2 0.75 = 0.748$

5-1.4 Independence

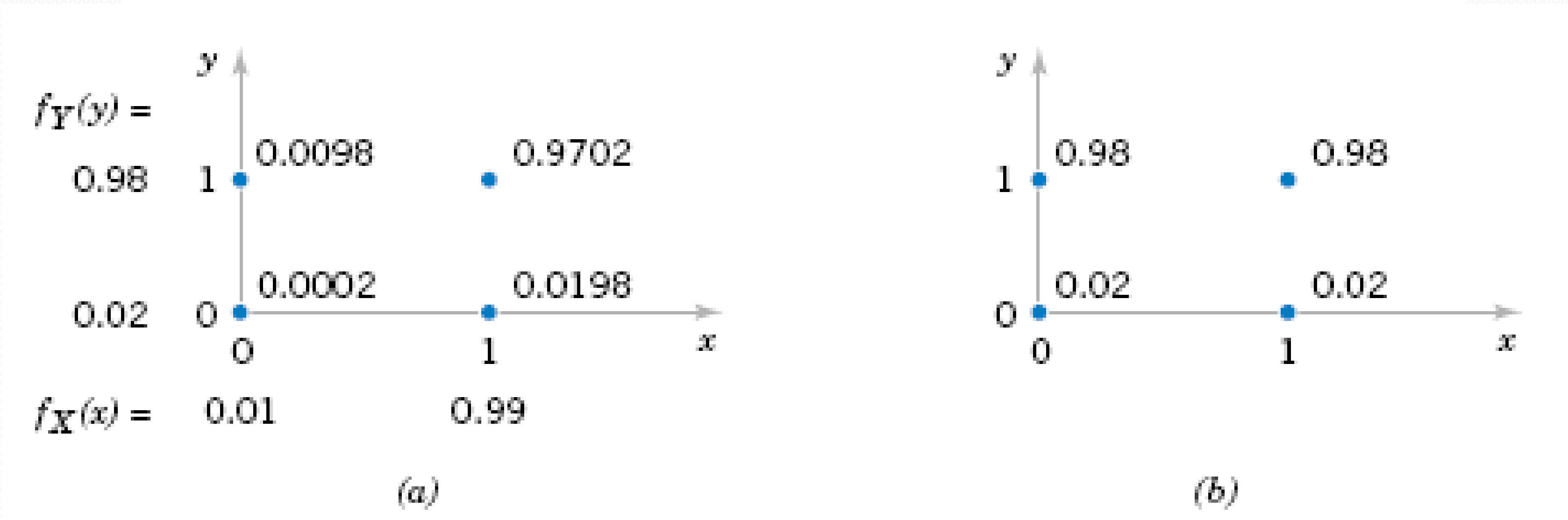
❖ In some random experiments, knowledge of the values of X does not change any of the probabilities associated with the values for Y.

➢ **Example:**

In a plastic molding operation, each part is classified as to whether it conforms to color and length specifications.

$$X = \begin{cases} 1 & \text{if the part conforms to color specifications} \\ 0 & \text{otherwise} \end{cases}$$

$$Y = \begin{cases} 1 & \text{if the part conforms to length specifications} \\ 0 & \text{otherwise} \end{cases}$$



Joint and marginal probability distributions of *X* and *Y*

Conditional probability distributions of *Y* given X=x

❖ Notice the effect of conditions…

❖ By analogy with independent events, we define two random variables to be **independent** whenever

$$f_{XY}(x, y) = f_X(x)\, f_Y(y)$$

For all x and y.

$$f_{Y|x}(y) = \frac{f_{XY}(x,y)}{f_X(x)} = \frac{f_X(x)\, f_Y(y)}{f_X(x)} = f_Y(y)$$

For discrete random variables X and Y, if any one of the following properties is true, the others are also true, and X and Y are **independent.**

(1) $f_{XY}(x, y) = f_X(x) f_Y(y)$ for all x and y

(2) $f_{Y|x}(y) = f_Y(y)$ for all x and y with $f_X(x) > 0$

(3) $f_{X|y}(x) = f_X(x)$ for all x and y with $f_Y(y) > 0$

(4) $P(X \in A, Y \in B) = P(X \in A)P(Y \in B)$ for any sets A and B in the range of X and Y, respectively.

➤ Question 5-3.

Show that the following function satisfies the properties of a joint probability mass function

x	y	$f_{XY}(x, y)$
−1	−2	1/8
−0.5	−1	1/4
0.5	1	1/2
1	2	1/8

$f(x,y) \geq 0 \ and \ \sum_{R} f(x,y) = 1$ ➡ Yes it is

a) $P(X < 0.5, Y < 1.5) = f_{XY}(-1,-2) + f_{XY}(-0.5,-1) = \frac{1}{8} + \frac{1}{4} = \frac{3}{8}$

b) $P(X < 0.5) = f_{XY}(-1,-2) + f_{XY}(-0.5,-1) = \frac{3}{8}$

c) $P(Y < 1.5) = f_{XY}(-1,-2) + f_{XY}(-0.5,-1) + f_{XY}(0.5,1) = \frac{7}{8}$

d) $P(X > 0.25, Y < 4.5) = f_{XY}(0.5,1) + f_{XY}(1,2) = \frac{5}{8}$

e)

$$E(X) = -1(\tfrac{1}{8}) - 0.5(\tfrac{1}{4}) + 0.5(\tfrac{1}{2}) + 1(\tfrac{1}{8}) = \tfrac{1}{8}$$

$$E(Y) = -2(\tfrac{1}{8}) - 1(\tfrac{1}{4}) + 1(\tfrac{1}{2}) + 2(\tfrac{1}{8}) = \tfrac{1}{4}$$

$$V(X)=(-1-1/8)^2(1/8)+(-0.5-1/8)^2(1/4)+(0.5-1/8)^2(1/2)+(1-1/8)^2(1/8)=0.4219$$

$$V(Y)=(-2-1/4)^2(1/8)+(-1-1/4)^2(1/4)+(1-1/4)^2(1/2)+(2-1/4)^2(1/8)=1.6875$$

f) marginal distribution of X

x	$f_X(x)$
-1	1/8
-0.5	¼
0.5	½
1	1/8

g) $f_{Y|X}(y) = \dfrac{f_{XY}(1, y)}{f_X(1)}$

y	$f_{Y\|X}(y)$
2	1/8/(1/8)=1

h) $f_{X|Y}(x) = \dfrac{f_{XY}(x,1)}{f_Y(1)}$

x	$f_{X\|Y}(x)$
0.5	½/(1/2)=1

i) $E(X|Y=1) = 0.5$

j) no, X and Y are not independent

5-2 TWO CONTINUOUS RANDOM VARIABLES:

5-2.1 Joint Probability Distributions

- **Example:**

let the continuous random variable X denote the length of one dimension of an injection-molded part, and let the continuous random variable Y denote the length of another dimension.

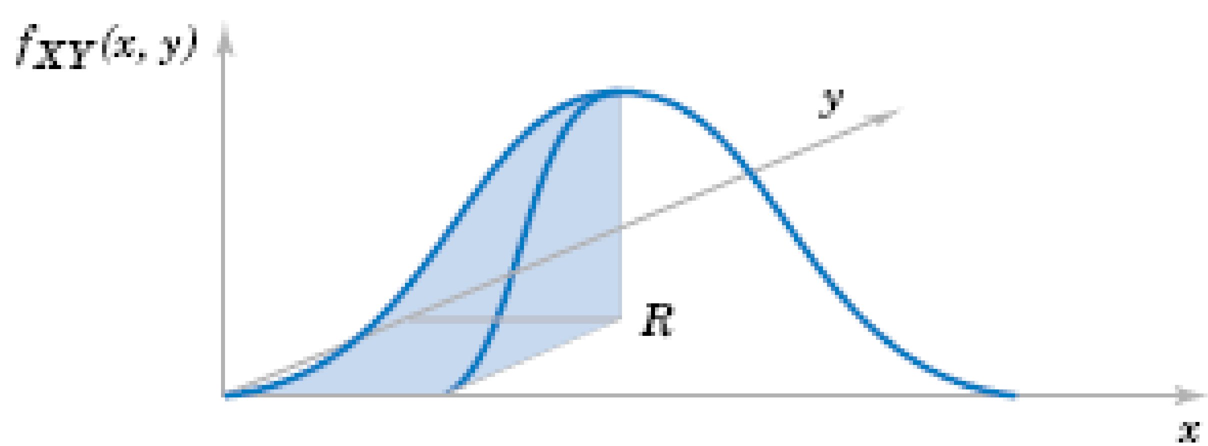
We can study each random variable separately.

However, because the two random variables are measurements from the same part, small disturbances in the injection-molding process, such as pressure and temperature variations, might be more likely to generate values for X and Y in specific regions of two-dimensional space.

Knowledge of the joint probability distribution of X and Y provides information that is not obvious from the marginal probability distributions.

❖ The joint probability distribution of two continuous random variables X and Y can be specified by providing a method for calculating the probability that X and Y assume a value in any region R of two-dimensional space.

A **joint probability density function** can be defined over two-dimensional space.



❖ The probability That (X,Y) assumes a value in the region R equals the volume of the shaded region

❖ Joint Probability Density Function:

A **joint probability density function** for the continuous random variables X and Y, denoted as $f_{XY}(x, y)$, satisfies the following properties:

(1) $f_{XY}(x, y) \geq 0$ for all x, y

(2) $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{XY}(x, y)\, dx\, dy = 1$

(3) For any region R of two-dimensional space

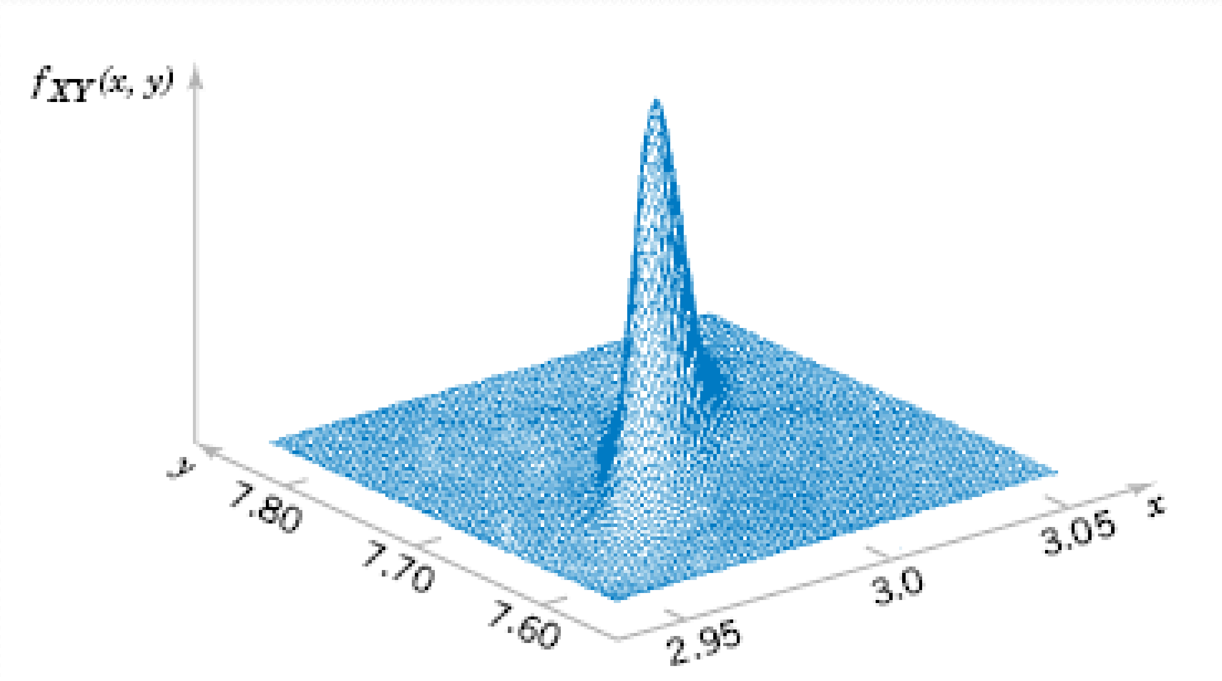
$$P([X, Y] \in R) = \iint_R f_{XY}(x, y)\, dx\, dy$$

- **Example:**

Each length might be modeled by a normal distribution. However, because the measurements are from the same part, the random variables are typically not independent.

If the specifications for X and Y are 2.95 to 3.05 and 7.60 to 7.80 millimeters, respectively, we might be interested in the probability that a part satisfies both specifications; that is,

$$P(2.95 < X < 3.05, 7.60 < Y < 7.80).$$

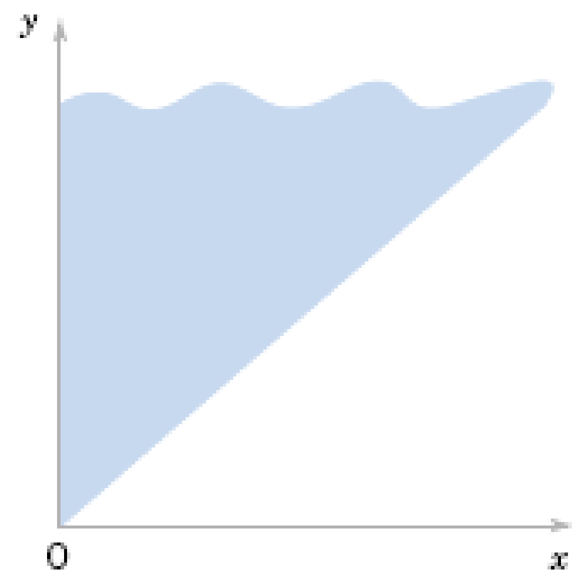


➢ **Example:**

Let the random variable X denote the time until a computer server connects to your machine (in milliseconds), and let Y denote the time until the server authorizes you as a valid user (in milliseconds).

$$f_{XY}(x, y) = 6 \times 10^{-6} \exp(-0.001x - 0.002y) \quad \text{for } x < y$$

The region with nonzero probability is shaded as follows.



The property that this joint probability density function integrates to 1 can be verified as follows:

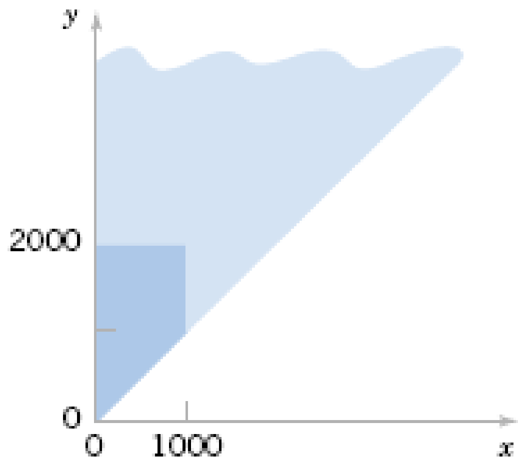
$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XY}(x,y)\,dy\,dx = \int_{0}^{\infty}\left(\int_{x}^{\infty} 6\times 10^{-6} e^{-0.001x-0.002y}\,dy\right)dx$$

$$= 6\times 10^{-6}\int_{0}^{\infty}\left(\int_{x}^{\infty} e^{-0.002y}\,dy\right)e^{-0.001x}\,dx$$

$$= 6\times 10^{-6}\int_{0}^{\infty}\left(\frac{e^{-0.002x}}{0.002}\right)e^{-0.001x}\,dx$$

$$= 0.003\left(\int_{0}^{\infty} e^{-0.003x}\,dx\right) = 0.003\left(\frac{1}{0.003}\right) = 1$$

$$P(X \le 1000, Y \le 2000) = \int_0^{1000} \int_x^{2000} f_{XY}(x, y)\, dy\, dx$$



$$= 6 \times 10^{-6} \int_0^{1000} \left(\int_x^{2000} e^{-0.002y}\, dy \right) e^{-0.001x}\, dx$$

$$= 6 \times 10^{-6} \int_0^{1000} \left(\frac{e^{-0.002x} - e^{-4}}{0.002} \right) e^{-0.001x}\, dx$$

$$= 0.003 \int_0^{1000} e^{-0.003x} - e^{-4} e^{-0.001x}\, dx$$

$$= 0.003 \left[\left(\frac{1 - e^{-3}}{0.003} \right) - e^{-4} \left(\frac{1 - e^{-1}}{0.001} \right) \right]$$

$$= 0.003\,(316.738 - 11.578) = 0.915$$

5-2.2 Marginal Probability Distributions

❖ **Marginal Probability Density Function:**

If the joint probability density function of continuous random variables X and Y is $f_{XY}(x, y)$, the **marginal probability density functions** of X and Y are

$$f_X(x) = \int_{R_x} f_{XY}(x, y)\, dy \quad \text{and} \quad f_Y(y) = \int_{R_y} f_{XY}(x, y)\, dx$$

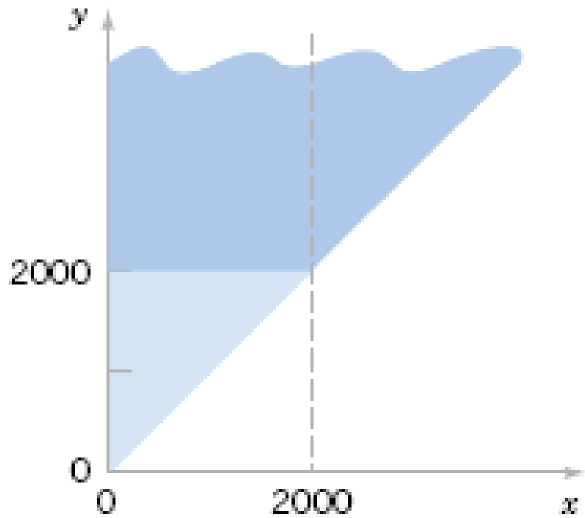
where R_x denotes the set of all points in the range of (X, Y) for which $X = x$ and R_y denotes the set of all points in the range of (X, Y) for which $Y = y$

❖ A probability involving only one random variable, can be found from the marginal probability distribution of X or from the joint probability distribution of X and Y.

$$P(a < X < b) = \int_a^b \int_{R_x} f_{XY}(x, y)\, dy\, dx = \int_a^b \left(\int_{R_x} f_{XY}(x, y)\, dy \right) dx = \int_a^b f_X(x)\, dx$$

➢ Example:

For the last example, calculate the probability that Y exceeds 2000 milliseconds.



$$P(Y > 2000) = \int_{0}^{2000} \left(\int_{2000}^{\infty} 6 \times 10^{-6} e^{-0.001x - 0.002y}\, dy \right) dx + \int_{2000}^{\infty} \left(\int_{x}^{\infty} 6 \times 10^{-6} e^{-0.001x - 0.002y}\, dy \right) dx$$

Alternatively, the probability can be calculated from the marginal probability distribution of Y as follows.

$$f_Y(y) = \int_0^y 6 \times 10^{-6} e^{-0.001x - 0.002y}\, dx = 6 \times 10^{-6} e^{-0.002y} \int_0^y e^{-0.001x}\, dx$$

$$= 6 \times 10^{-6} e^{-0.002y} \left(\left. \frac{e^{-0.001x}}{-0.001} \right|_0^y \right) = 6 \times 10^{-6} e^{-0.002y} \left(\frac{1 - e^{-0.001y}}{0.001} \right)$$

$$= 6 \times 10^{-3} e^{-0.002y} (1 - e^{-0.001y}) \quad \text{for } y > 0$$

$$P(Y > 2000) = 6 \times 10^{-3} \int_{2000}^{\infty} e^{-0.002y} (1 - e^{-0.001y})\, dy$$

$$= 6 \times 10^{-3} \left[\left(\left. \frac{e^{-0.002y}}{-0.002} \right|_{2000}^{\infty} \right) - \left(\left. \frac{e^{-0.003y}}{-0.003} \right|_{2000}^{\infty} \right) \right]$$

$$= 6 \times 10^{-3} \left[\frac{e^{-4}}{0.002} - \frac{e^{-6}}{0.003} \right] = 0.05$$

5-2.3 Conditional Probability Distributions

Given continuous random variables X and Y with joint probability density function $f_{XY}(x, y)$, the **conditional probability density function** of Y given $X = x$ is

$$f_{Y|x}(y) = \frac{f_{XY}(x, y)}{f_X(x)} \quad \text{for} \quad f_X(x) > 0$$

Because the conditional probability density function $f_{Y|x}(y)$ is a probability density function for all y in R_x, the following properties are satisfied:

(1) $f_{Y|x}(y) \geq 0$

(2) $\int_{R_x} f_{Y|x}(y)\,dy = 1$

(3) $P(Y \in B | X = x) = \int_B f_{Y|x}(y)\,dy$ for any set B in the range of Y

➢ **Example:**

Determine the conditional probability density function for Y given that X=x.

$$f_X(x) = \int_x^{\infty} 6 \times 10^{-6} e^{-0.001x-0.002y}\, dy = 6 \times 10^{-6} e^{-0.001x} \left(\frac{e^{-0.002y}}{-0.002} \bigg|_x^{\infty} \right)$$

$$= 6 \times 10^{-6} e^{-0.001x} \left(\frac{e^{-0.002x}}{0.002} \right) = 0.003 e^{-0.003x} \quad \text{for} \quad x > 0$$

$$f_{Y|x}(y) = f_{XY}(x, y)/f_x(x) = \frac{6 \times 10^{-6} e^{-0.001x-0.002y}}{0.003 e^{-0.003x}}$$

$$= 0.002 e^{0.002x-0.002y} \quad \text{for } 0 < x \quad \text{and} \quad x < y$$

$$P(Y > 2000 | x = 1500) = \int_{2000}^{\infty} f_{Y|1500}(y)\, dy = \int_{2000}^{\infty} 0.002 e^{0.002(1500)-0.002y}\, dy$$

$$= 0.002 e^3 \left(\frac{e^{-0.002y}}{-0.002} \bigg|_{2000}^{\infty} \right) = 0.002 e^3 \left(\frac{e^{-4}}{0.002} \right) = 0.368$$

❖ Conditional Mean and Variance:

Let R_x denote the set of all points in the range of (X, Y) for which $X = x$. The **conditional mean** of Y given $X = x$, denoted as $E(Y|x)$ or $\mu_{Y|x}$, is

$$E(Y|x) = \int_{R_x} y f_{Y|x}(y)\, dy$$

and the **conditional variance** of Y given $X = x$, denoted as $V(Y|x)$ or $\sigma^2_{Y|x}$, is

$$V(Y|x) = \int_{R_x} (y - \mu_{Y|x})^2 f_{Y|x}(y)\, dy = \int_{R_x} y^2 f_{Y|x}(y)\, dy - \mu^2_{Y|x}$$

➢ Example:

Determine the conditional mean for Y given that x = 1500.

$$E(Y|x=1500) = \int_{1500}^{\infty} y(0.002e^{0.002(1500)-0.002y})\,dy = 0.002e^{3}\int_{1500}^{\infty} ye^{-0.002y}\,dy$$

Integrate by parts as follows:

$$\int_{1500}^{\infty} ye^{-0.002y}\,dy = y\frac{e^{-0.002y}}{-0.002}\Big|_{1500}^{\infty} - \int_{1500}^{\infty}\left(\frac{e^{-0.002y}}{-0.002}\right)dy$$

$$= \frac{1500}{0.002}e^{-3} - \left(\frac{e^{-0.002y}}{(-0.002)(-0.002)}\Big|_{1500}^{\infty}\right)$$

$$= \frac{1500}{0.002}e^{-3} + \frac{e^{-3}}{(0.002)(0.002)} = \frac{e^{-3}}{0.002}(2000)$$

With the constant $0.002e^{3}$ reapplied

$$E(Y|x=1500) = 2000$$

5-2.4 Independence

❖ The definition of independence for continuous random variables is similar to the definition for discrete random variables.

For continuous random variables X and Y, if any one of the following properties is true, the others are also true, and X and Y are said to be **independent.**

(1) $f_{XY}(x, y) = f_X(x) f_Y(y)$ for all x and y

(2) $f_{Y|x}(y) = f_Y(y)$ for all x and y with $f_X(x) > 0$

(3) $f_{X|y}(x) = f_X(x)$ for all x and y with $f_Y(y) > 0$

(4) $P(X \in A, Y \in B) = P(X \in A)P(Y \in B)$ for any sets A and B in the range of X and Y, respectively.

➢ **Example:**

Suppose that,

$$f_{XY}(x, y) = 2 \times 10^{-6} e^{-0.001x - 0.002y} \text{ for } x \geq 0 \text{ and } y \geq 0.$$

Then,

$$f_X(x) = \int_0^{\infty} 2 \times 10^{-6} e^{-0.001x - 0.002y}\, dy = 0.001\, e^{-0.001x} \quad \text{for } x > 0$$

$$f_Y(y) = \int_0^{\infty} 2 \times 10^{-6} e^{-0.001x - 0.002y}\, dx = 0.002\, e^{-0.002y} \quad \text{for } y > 0$$

Since the multiplication of marginal probability functions of X and Y is equal to the original joint probability density function, then X and Y are **INDEPENDENT..**

➢ Question 5-18.

Determine the value or (c) that makes f(x,y) = c (x+y) a joint probability density function over the range 0 < x < 3 and x < y < x+2.

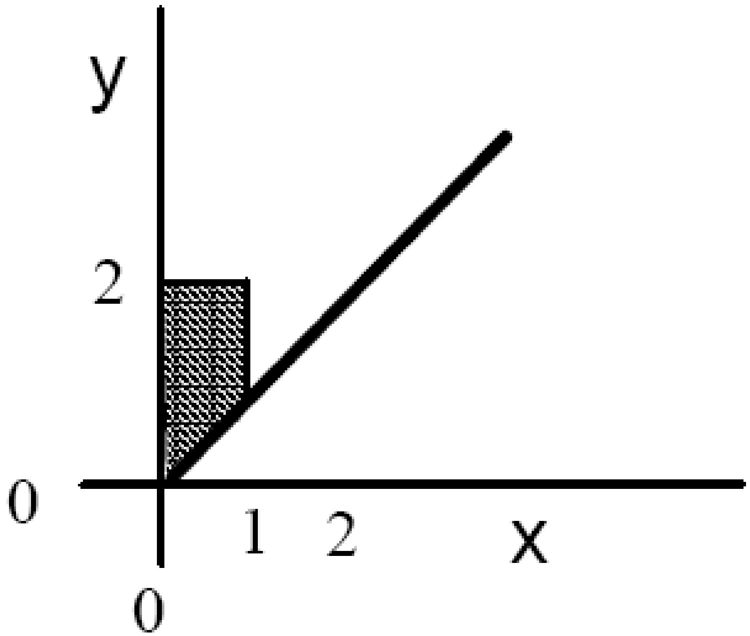
$$c\int_0^3 \int_x^{x+2} (x+y)\,dy\,dx = \int_0^3 xy + \frac{y^2}{2}\Bigg|_x^{x+2} dx$$

$$= \int_0^3 \left[x(x+2) + \frac{(x+2)^2}{2} - x^2 - \frac{x^2}{2}\right]dx$$

$$= c\int_0^3 (4x+2)\,dx = \left[2x^2 + 2x\right]_0^3 = 24c$$

Therefore, c = 1/24.

a) P(X < 1, Y < 2) equals the integral of $f_{XY}(x, y)$ over the following region.

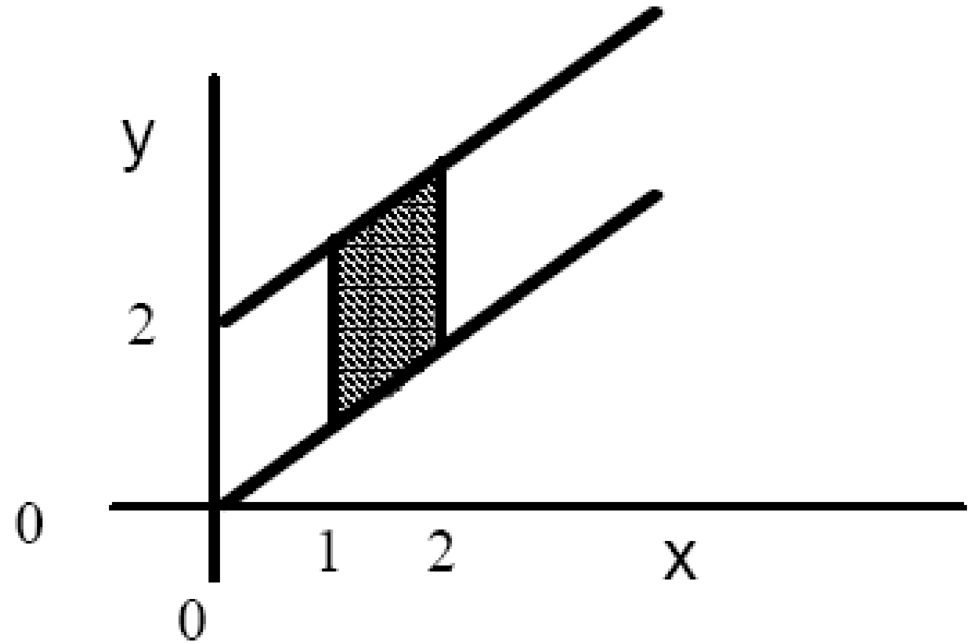


Then,

$$P(X<1,Y<2)=\frac{1}{24}\int_0^1\int_x^2 (x+y)dydx=\frac{1}{24}\int_0^1 xy+\frac{y^2}{2}\Big|_x^2\,dx=\frac{1}{24}\int_0^3 2x+2-\frac{3x^2}{2}dx=$$

$$\frac{1}{24}\left[x^2+2x-\frac{x^3}{2}\Big|_0^1\right]=0.10417$$

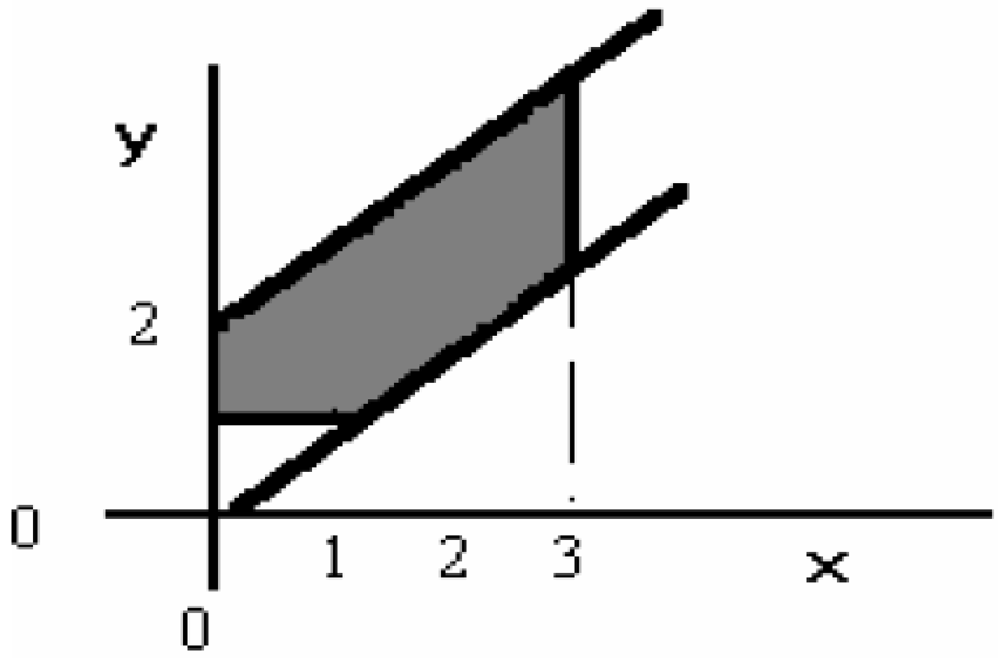
b) P(1 < X < 2) equals the integral of $f_{XY}(x, y)$ over the following region.



$$P(1 < X < 2) = \frac{1}{24}\int_1^2 \int_x^{x+2} (x+y)\,dy\,dx = \frac{1}{24}\int_1^2 xy + \frac{y^2}{2}\Big|_x^{x+2} dx$$

$$= \frac{1}{24}\int_0^3 (4x+2)\,dx = \frac{1}{24}\left[2x^2 + 2x\Big|_1^2\right] = \tfrac{1}{6}.$$

c) P(Y > 1) is the integral of $f_{XY}(x, y)$ over the following region.

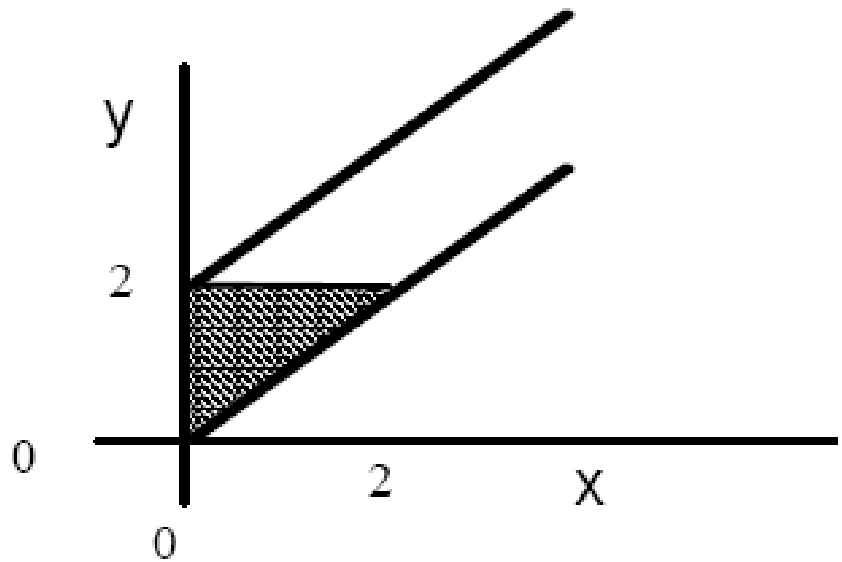


$$P(Y>1)=1-P(Y\le 1)=1-\tfrac{1}{24}\int_0^1\int_x^1 (x+y)dydx = 1-\frac{1}{24}\int_0^1 (xy+\frac{y^2}{2})\Bigg|_x^1$$

$$=1-\frac{1}{24}\int_0^1 x+\frac{1}{2}-\frac{3}{2}x^2 dx = 1-\frac{1}{24}\left(\frac{x^2}{2}+\frac{1}{2}-\frac{1}{2}x^3\right)\Bigg|_0^1$$

$$=1-0.02083=0.9792$$

d) P(X < 2, Y < 2) is the integral of $f_{XY}(x,y)$ over the following region.



$$E(X) = \frac{1}{24}\int_0^3 \int_x^{x+2} x(x+y)\,dy\,dx = \frac{1}{24}\int_0^3 x^2 y + \frac{xy^2}{2}\Big|_x^{x+2} dx$$

$$= \frac{1}{24}\int_0^3 (4x^2 + 2x)\,dx = \frac{1}{24}\left[\frac{4x^3}{3} + x^2\Big|_0^3\right] = \frac{15}{8}$$

e)

$$E(X)=\frac{1}{24}\int_0^3\int_x^{x+2}x(x+y)dydx=\frac{1}{24}\int_0^3 x^2y+\frac{xy^2}{2}\Big|_x^{x+2}dx$$

$$=\frac{1}{24}\int_0^3(4x^2+2x)dx=\frac{1}{24}\left[\frac{4x^3}{3}+x^2\Big|_0^3\right]=\frac{15}{8}$$

f)

$$V(X)=\frac{1}{24}\int_0^3\int_x^{x+2}x^2(x+y)dydx-\left(\frac{15}{8}\right)^2=\frac{1}{24}\int_0^3 x^3y+\frac{x^2y^2}{2}\Big|_x^{x+2}dx-\left(\frac{15}{8}\right)^2$$

$$=\frac{1}{24}\int_0^3(3x^3+4x^2+4x-\frac{x^4}{4})dx-\left(\frac{15}{8}\right)^2$$

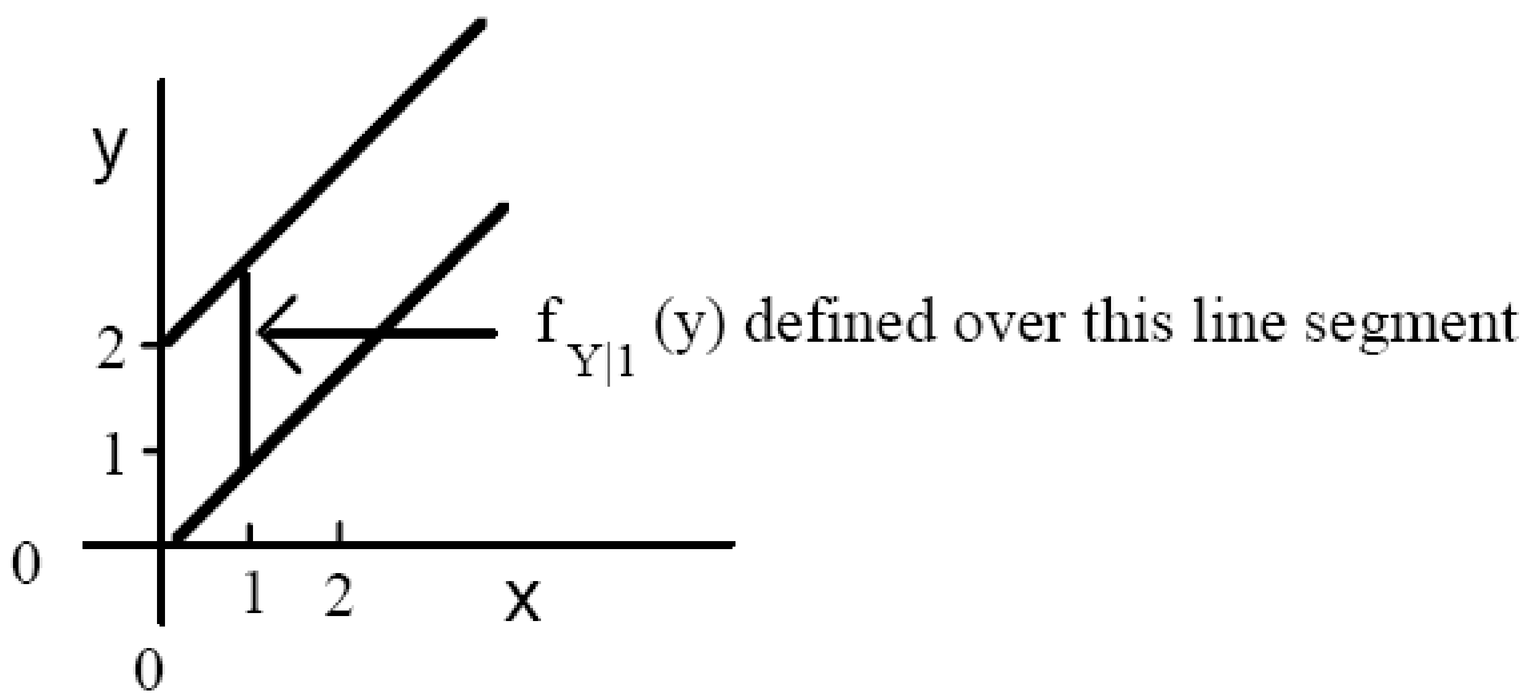
$$=\frac{1}{24}\left[\frac{3x^4}{4}+\frac{4x^3}{3}+2x^2-\frac{x^5}{20}\Big|_0^3\right]-\left(\frac{15}{8}\right)^2=\frac{31707}{320}$$

g) $f_X(x)$ is the integral of $f_{XY}(x,y)$ over the interval from x to x+2. That is,

$$f_X(x) = \frac{1}{24}\int_x^{x+2}(x+y)dy = \frac{1}{24}\left[xy + \frac{y^2}{2}\bigg|_x^{x+2}\right] = \frac{x}{6} + \frac{1}{12} \quad \text{for } 0 < x < 3.$$

h) $f_{Y|1}(y) = \frac{f_{XY}(1,y)}{f_X(1)} = \frac{\frac{1}{24}(1+y)}{\frac{1}{6}+\frac{1}{12}} = \frac{1+y}{6}$ for $1 < y < 3$.

See the following graph,

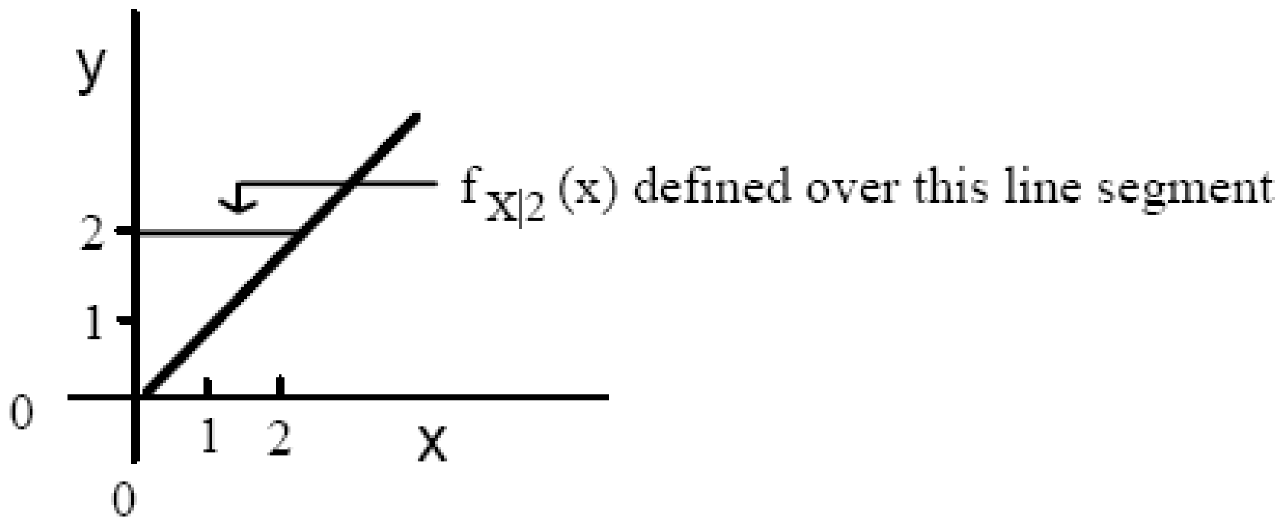


i) E(Y|X=1) = $\int_1^3 y\left(\frac{1+y}{6}\right)dy = \frac{1}{6}\int_1^3 (y+y^2)dy = \frac{1}{6}\left(\frac{y^2}{2}+\frac{y^3}{3}\right)\Bigg|_1^3$ =2.111

j) $P(Y>2 \mid X=1) = \int_2^3 \left(\frac{1+y}{6}\right)dy = \frac{1}{6}\int_2^3 (1+y)dy = \frac{1}{6}\left(y+\frac{y^2}{2}\right)\Bigg|_2^3$ =0.5833

k) $f_{X|2}(x) = \frac{f_{XY}(x,2)}{f_Y(2)}$. Here $f_Y(y)$ is determined by integrating over x. There are three regions of integration. For $0 < y \le 2$ the integration is from 0 to y. For $2 < y \le 3$ the integration is from y-2 to y. For $3 < y < 5$ the integration is from y to 3. Because the condition is y=2, only the first integration is needed.

$$f_Y(y) = \frac{1}{24}\int_0^y (x+y)dx = \frac{1}{24}\left[\frac{x^2}{2}+xy\Bigg|_0^y\right] = \frac{y^2}{16} \quad \text{for } 0 < y \le 2 .$$



Therefore, $f_Y(2) = 1/4$ and $f_{X|2}(x) = \dfrac{\frac{1}{24}(x+2)}{1/4} = \dfrac{x+2}{6}$ for 0 < x < 3

5-3 COVARIANCE AND CORRELATION:

❖ A common measure of the relationship between two random variables is the **covariance**.

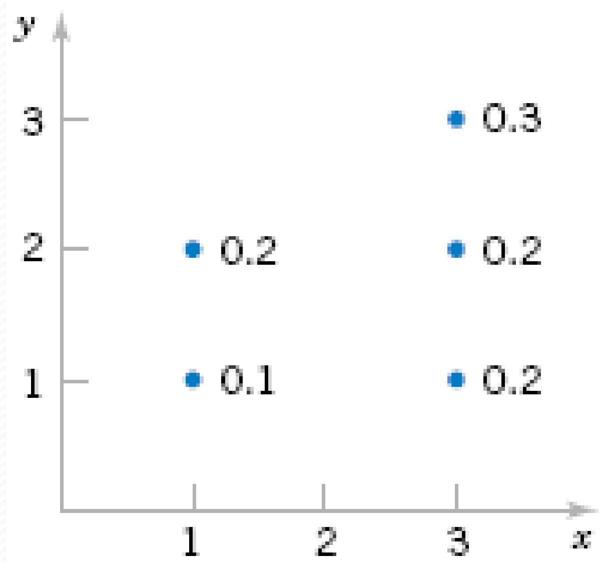
❖ To define the covariance, we need to describe the expected value of a function of two random variables *h(X, Y)*.

❖ **Expected Value of a Function of Two Random Variables:**

$$E[h(X,Y)] = \begin{cases} \sum\sum_{R} h(x,y) f_{XY}(x,y) & X, Y \text{ discrete} \\ \iint_{R} h(x,y) f_{XY}(x,y)\,dx\,dy & X, Y \text{ continuous} \end{cases}$$

➢ **Example:**

For the following joint probability distribution, Calculate $E[(X-\mu_X)(Y-\mu_Y)]$.



$$\mu_X = 1 \times 0.3 + 3 \times 0.7 = 2.4$$

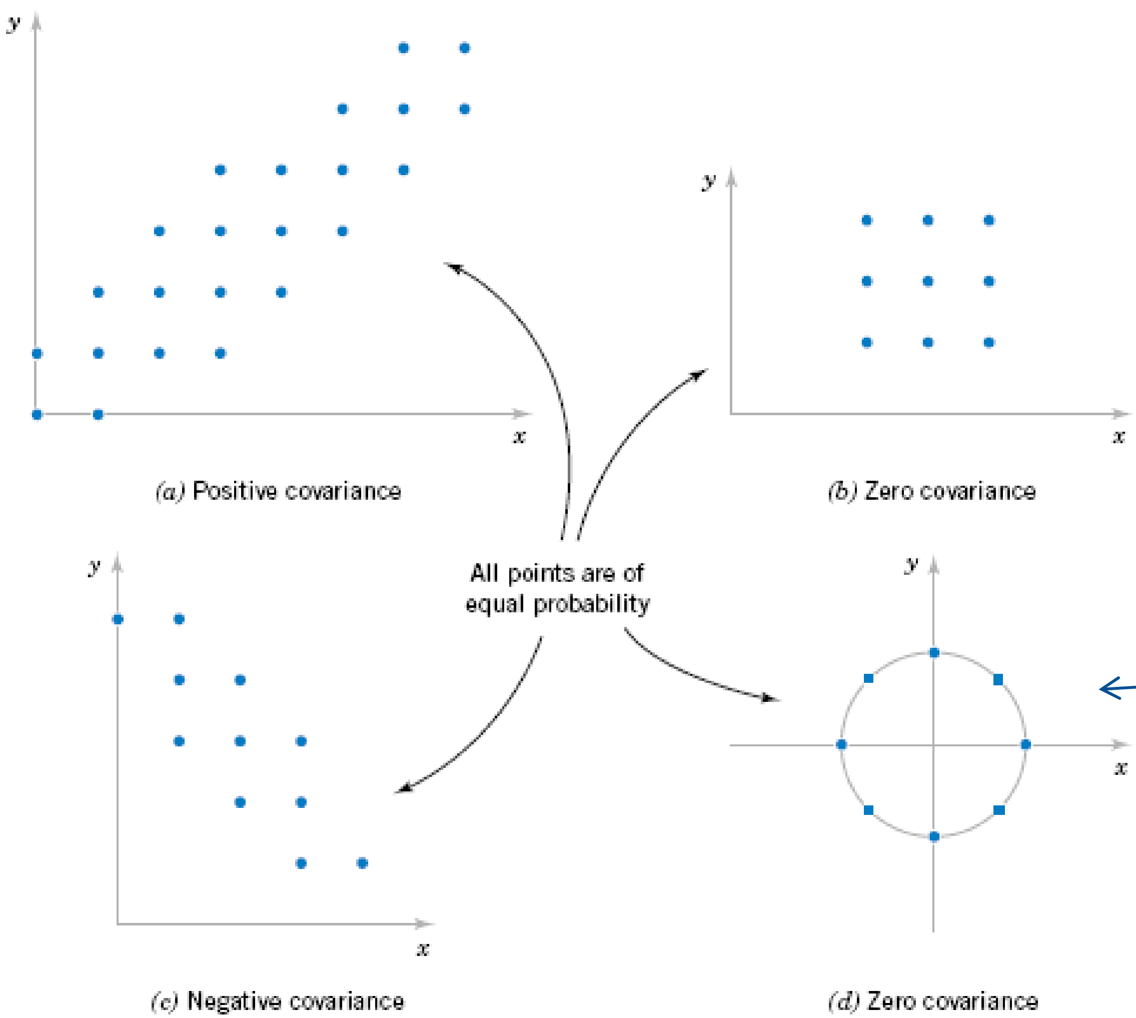
$$\mu_Y = 1 \times 0.3 + 2 \times 0.4 + 3 \times 0.3 = 2.0$$

$$\begin{aligned} E[(X-\mu_X)(Y-\mu_Y)] &= (1-2.4)(1-2.0) \times 0.1 \\ &+ (1-2.4)(2-2.0) \times 0.2 + (3-2.4)(1-2.0) \times 0.2 \\ &+ (3-2.4)(2-2.0) \times 0.2 + (3-2.4)(3-2.0) \times 0.3 = 0.2 \end{aligned}$$

❖Covariance

The **covariance** between the random variables X and Y, denoted as $\text{cov}(X, Y)$ or σ_{XY}, is

$$\sigma_{XY} = E[(X - \mu_X)(Y - \mu_Y)] = E(XY) - \mu_X \mu_Y$$



The covariance between X and Y describes the variation between the two random variables.

Covariance is a measure of **linear relationship between the random variables. If the relationship** between the random variables is nonlinear, the covariance might not be sensitive to the relationship.

❖Correlation

❖ It is another measure of the relationship between two random variables that is often easier to interpret than the covariance.

The **correlation** between random variables X and Y, denoted as ρ_{XY}, is

$$\rho_{XY} = \frac{\operatorname{cov}(X, Y)}{\sqrt{V(X)V(Y)}} = \frac{\sigma_{XY}}{\sigma_X \sigma_Y}$$

For any two random variables X and Y

$$-1 \leq \rho_{XY} \leq +1$$

❖ The correlation just scales the covariance by the standard deviation of each variable.

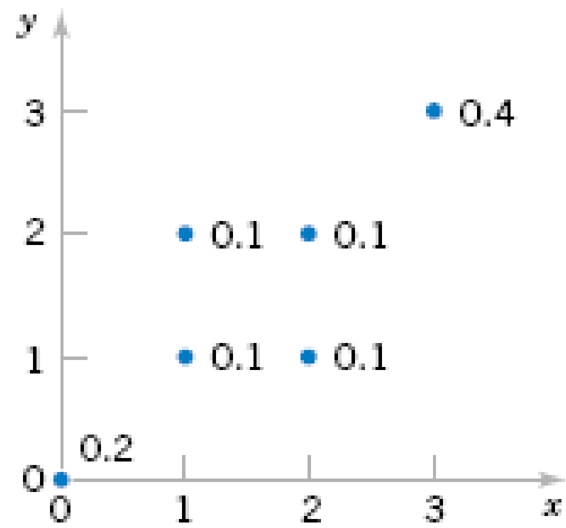
❖ The correlation is a dimensionless quantity that can be used to compare the linear relationships between pairs of variables in different units.

❖ If the points in the joint probability distribution of X and Y that receive positive probability tend to fall along a line of positive (or negative) slope, the correlation is near +1 (or -1).

❖ If it is equal to +1 or -1 , it can be shown that the points in the joint probability distribution that receive positive probability fall exactly along a straight line.

❖ Two random variables with nonzero correlation are said **to be correlated**. Similar to covariance, the correlation is a measure of the linear relationship between random variables.

- **Example:**



$$E(XY) = 0 \times 0 \times 0.2 + 1 \times 1 \times 0.1 + 1 \times 2 \times 0.1 + 2 \times 1 \times 0.1 + 2 \times 2 \times 0.1 + 3 \times 3 \times 0.4 = 4.5$$

$$E(X) = 0 \times 0.2 + 1 \times 0.2 + 2 \times 0.2 + 3 \times 0.4 = 1.8$$

$$V(X) = (0 - 1.8)^2 \times 0.2 + (1 - 1.8)^2 \times 0.2 + (2 - 1.8)^2 \times 0.2 + (3 - 1.8)^2 \times 0.4 = 1.36$$

$$\sigma_{XY} = E(XY) - E(X)E(Y) = 4.5 - (1.8)(1.8) = 1.26$$

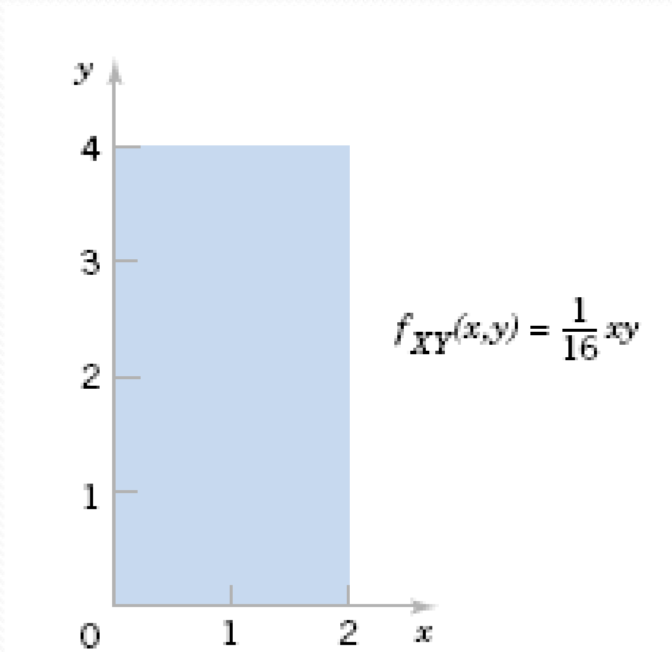
$$\rho_{XY} = \frac{\sigma_{XY}}{\sigma_X \sigma_Y} = \frac{1.26}{(\sqrt{1.36})(\sqrt{1.36})} = 0.926$$

❖ For independent random variables, we do not expect any relationship in their joint probability distribution.

If X and Y are independent random variables,

$$\sigma_{XY} = \rho_{XY} = 0$$

➢ **Example:**



$$E(XY) = \int_0^4 \int_0^2 xy f_{XY}(x, y)\,dx\,dy = \frac{1}{16}\int_0^4 \left[\int_0^2 x^2 y^2\,dx\right] dy = \frac{1}{16}\int_0^4 y^2 \left[x^3/3 \Big|_0^2 \right]$$

$$= \frac{1}{16}\int_0^4 y^2 [8/3]\,dy = \frac{1}{6}\left[y^3/3 \Big|_0^4 \right] = \frac{1}{6}[64/3] = 32/9$$

$$E(X) = \int_0^4 \int_0^2 x f_{XY}(x, y)\,dx\,dy = \frac{1}{16}\int_0^4 \left[\int_0^2 x^2\,dx\right] dy = \frac{1}{16}\int_0^4 \left[x^3/3 \Big|_0^2 \right] dy$$

$$= \frac{1}{16}\left[y^2/2 \Big|_0^4 \right][8/3] = \frac{1}{6}[16/2] = 4/3$$

$$E(Y) = \int_0^4 \int_0^2 y f_{XY}(x, y)\,dx\,dy = \frac{1}{16}\int_0^4 y^2 \left[\int_0^2 x\,dx\right] dy = \frac{1}{16}\int_0^4 y^2 \left[x^2/2 \Big|_0^2 \right] dy$$

$$= \frac{2}{16}\left[y^3/3 \Big|_0^4 \right] = \frac{1}{8}[64/3] = 8/3$$

$$E(XY) - E(X)E(Y) = 32/9 - (4/3)(8/3) = 0$$

It can be shown that these two random variables are independent.

❖ However, if the correlation between two random variables is zero, we cannot immediately conclude that the random variables are independent.

➢ **Question 5-35.**

Determine the value or (c) and the covariance and correlation for the joint probability mass function f(x,y) = c (x+y) for x = 1,2,3 and y = 1,2,3.

$$\sum_{x=1}^{3}\sum_{y=1}^{3} c(x+y) = 36c, \qquad c = 1/36$$

$$E(X) = \frac{13}{6} \qquad E(Y) = \frac{13}{6} \qquad E(XY) = \frac{14}{3} \qquad \sigma_{xy} = \frac{14}{3} - \left(\frac{13}{6}\right)^2 = \frac{-1}{36}$$

$$E(X^2) = \frac{16}{3} \qquad E(Y^2) = \frac{16}{3} \qquad V(X) = V(Y) = \frac{23}{36}$$

$$\rho = \frac{\frac{-1}{36}}{\sqrt{\frac{23}{36}}\sqrt{\frac{23}{36}}} = -0.0435$$

5-5 LINEAR FUNCTIONS OF RANDOM VARIABLES:

❖ A random variable is sometimes defined as a function of one or more random variables.

➢ **Example:**

If the random variables X_1 and X_2 denote the length and width, respectively, of a manufactured part, $Y = 2X_1 + 2X_2$ is a random variable that represents the perimeter of the part.

❖ In this section, we develop results for random variables that are linear combinations of random variables.

❖**Linear Combination:**

Given random variables $X_1, X_2, \ldots, X_p$ and constants $c_1, c_2, \ldots, c_p$,

$$Y = c_1X_1 + c_2X_2 + \cdots + c_pX_p$$

is a **linear combination** of $X_1, X_2, \ldots, X_p$.

❖ Mean of a Linear Function:

If $Y = c_1X_1 + c_2X_2 + \cdots + c_pX_p$,

$$E(Y) = c_1E(X_1) + c_2E(X_2) + \cdots + c_pE(X_p)$$

❖ Variance of a Linear Function:

If $X_1, X_2, \ldots, X_p$ are random variables, and $Y = c_1X_1 + c_2X_2 + \cdots + c_pX_p$, then in general

$$V(Y) = c_1^2V(X_1) + c_2^2V(X_2) + \cdots + c_p^2V(X_p) + 2\sum_{i<j}\sum c_ic_j\,\mathrm{cov}(X_i, X_j)$$

If $X_1, X_2, \ldots, X_p$ are **independent**,

$$V(Y) = c_1^2V(X_1) + c_2^2V(X_2) + \cdots + c_p^2V(X_p)$$

- This requires the random variables to be independent.

- **Example:**

A semiconductor product consists of three layers. If the variances in thickness of the first, second, and third layers are 25, 40, and 30 nanometers squared, what is the variance of the thickness of the final product?

$$X = X_1 + X_2 + X_3$$

So that

$$V(X) = V(X_1) + V(X_2) + V(X_3) = 25{+}40{+}30 = 95 \text{ nanometers squared.}$$

S.D. = 9.75 nm

❖ Mean and Variance of an Average:

If $\overline{X} = (X_1 + X_2 + \cdots + X_p)/p$ with $E(X_i) = \mu$ for $i = 1, 2, \ldots, p$

$$E(\overline{X}) = \mu$$

if $X_1, X_2, \ldots, X_p$ are also independent with $V(X_i) = \sigma^2$ for $i = 1, 2, \ldots, p$,

$$V(\overline{X}) = \frac{\sigma^2}{p}$$

❖ Reproductive Property of the Normal Distribution:

If $X_1, X_2, \ldots, X_p$ are independent, normal random variables with $E(X_i) = \mu_i$ and $V(X_i) = \sigma_i^2$, for $i = 1, 2, \ldots, p$,

$$Y = c_1X_1 + c_2X_2 + \cdots + c_pX_p$$

is a normal random variable with

$$E(Y) = c_1\mu_1 + c_2\mu_2 + \cdots + c_p\mu_p$$

and

$$V(Y) = c_1^2\sigma_1^2 + c_2^2\sigma_2^2 + \cdots + c_p^2\sigma_p^2$$

Example:

Let the random variables X_1 and X_2 denote the length and width, respectively, of a manufactured part. Assume that X_1 is normal with $E(X_1) = 2$ centimeters and standard deviation 0.1 centimeter and that X_2 is normal with $E(X_2) = 5$ centimeters and standard deviation 0.2 centimeter. Also, assume that X_1 and X_2 are independent. Determine the probability that the perimeter exceeds 14.5 centimeters.

The Perimeter $Y = 2X_1 + 2X_2$

$E(Y) = 2*2 + 2*5 = 14$ cm

$V(Y) = 4*0.1^2 + 4*0.2^2 = 0.2$

so that

$$P(Y > 14.5) = P[(Y - \mu_Y)/\sigma_Y > (14.5 - 14)/\sqrt{0.2}]$$
$$= P(Z > 1.12) = 0.13$$

➢ Question 5-64.

Assume that the weights of individuals are independent and normally distributed with mean of 160 pounds and a standard deviation of 30 pounds. Suppose that 25 people squeeze into an elevator that is designed to hold 43000 pounds.

A- what is the probability that the load (total weight) exceeds the design limit?

$X \sim N(160, 900)$

Let $Y = X_1 + X_2 + \ldots + X_{25}$, $E(Y) = 25E(X) = 4000$, $V(Y) = 25^2(900) = 22500$

$P(Y > 4300) =$

$$P\left(Z > \frac{4300 - 4000}{\sqrt{22500}}\right) = P(Z > 2) = 1 - P(Z < 2) = 1 - 0.9773 = 0.0227$$

B- what design limit is exceeded by 25 occupants with probability 0.0001??

$$P(Y > x) = 0.0001 \text{ implies that } P\left(Z > \frac{x - 4000}{\sqrt{22500}}\right) = 0.0001.$$

Then $\frac{x-4000}{150} = 3.72$ and $x = 4558$